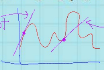
Introduction

- Calculus was invented (independently) by both Sir Isaac Newton and Gottfried Leibniz at the end of the 17th century.
- Calculus finds patterns between equations: it lets you answer questions like:
  - How does an equation grow & shrink?
  - When does it reach its highest/lowest point?
  - How do we use variables that are constantly changing (heat, motion, populations…)?
- Calculus is used in many different fields including engineering, economics, architecture, space exploration etc…

Differentiation

- Differentiation is a technique that calculates the slope of a tangent line at a particular point on a function.



- The slope of the tangent is useful because it tells us the "instantaneous rate of change" at that point.
- When we differentiate a function, we find the derivative. This is written as $\frac{dy}{dx}$ ("dee y by dee x") or $f'(x)$ ("f dashed x").

5km in 15 mins
20km/h

How to Differentiate

- Multiply the coefficient by the exponent and reduce the exponent by one.
- eg. $\boxed{f(x) = 2x^4 + 6x^3 + 2x^2}$

$$f'(x) = 2(4)x^{4-1} + 6(3)x^{3-1} + 2(2)x^{2-1}$$

$$\boxed{f'(x) = 8x^3 + 18x^2 + 4x}$$

- eg. $f(x) = 5x^3 - 3x + 4x^0$

numbers with no "x" differentiate to zero

$$f'(x) = 15x^2 - 3(1)x^0 + 4(0)$$

$$= 15x^2 - 3 + 0$$

$$= \boxed{15x^2 - 3}$$

Differentiating Fractions

- Functions that have x terms on the denominator are first rewritten so they have negative exponents.
- eg. $f(x) = 7x^2 + 5x + \frac{4}{x} - \frac{3}{x^3}$

$$f(x) = 7x^2 + 5x + 4x^{-1} - 3x^{-3}$$

$$f'(x) = 7(2)x + 5(1) + 4(-1)x^{-2} - 3(-3)x^{-4}$$

$$\boxed{f'(x) = 14x + 5 - 4x^{-2} + 9x^{-4}}$$

"write with no negative exponents

$$\rightarrow f'(x) = 14x + 5 - \frac{4}{x^2} + \frac{9}{x^4}$$

Worked Examples

1. Find f'(x) for $f(x) = \frac{x^2 + 4x - 5}{x}$

$$f(x) = \frac{x^2}{x} + \frac{4x}{x} - \frac{5}{x}$$

$$f(x) = x + 4 - 5x^{-1}$$

$$\boxed{f'(x) = 1 + 5x^{-2}}$$

$$x = 1x^1$$
$$= 1(1)x^0$$
$$= 1$$

2. If $y = 3x^2 - 4x$, find f'(x) and interpret it's meaning.

$$f'(x) = 6x - 4$$

this gives the slope of the original function at any point
& represents the instantaneous rate of change.

Gradient of the Tangent

- Differentiating gives us the equation for the gradient of the tangent to a function at any point.
- The slope / gradient of the tangent is the instantaneous rate of change.
- Eg If the function is the distance (or displacement) a car travels with time, the gradient of the tangent is the instantaneous rate of $\frac{\text{change in distance}}{\text{change in time}}$, or instaneous speed.

Finding the Gradient of a Tangent

• The gradient of a tangent at a point on a function is found by:

1. differentiate the function.
2. substitute the x-value of the point into the derivative. Answer = gradient of tangent

• eg. Find the slope of the tangent of $\boxed{f(x) = 3x^2 - 2x + 1}$ at (1,2).

$x = 1$

① $f'(x) = 6x - 2$

$f'(1) = 6(1) - 2$

$= 4$

Using the Calculator Ti 84

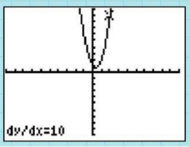
Finding the gradient to a curve at a point: eg Find the gradient to the curve $y = 3x^2 - 2x + 1$ at the point where x = 2.

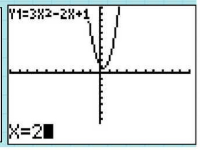
Press Y= and type in the equation of the curve.

Press 2nd TRACE 6 (dy/dx).

Type 2 (for $x = 2$) ENTER.

The answer is given under the graph.





Using the Calculator Nspire

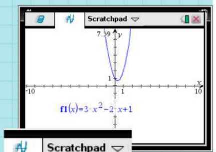
Finding the gradient to a curve at a point: eg Find the gradient to the curve $y = 3x^2 - 2x + 1$ at the point where x = 2.

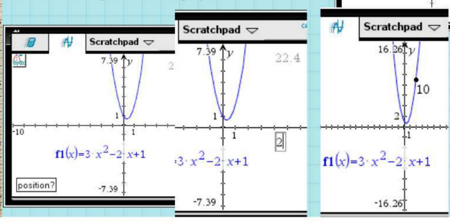
Open a graph scratchpad and type in the equation of the curve.

Press MENU; 6: Analyse graph; 5:dy/dx

Type 2 (for $x = 2$) ENTER.

The answer is given on the graph.





Worked Examples

4. Find f′ (x) for f(x) = $x^2 - 4x^{-1}$ and hence find the gradient of the tangent to the function at $x = 2$.

$f'(x) = 2x + 4x^{-2}$

$f'(2) = 2(2) + 4(2)^{-2}$

$= 4 + \frac{4}{2^2}$

$= \boxed{5}$

5. Find the gradient to the curve $\boxed{y = x^2 - 2x + 3x^{-1}}$ at the point where $x = 2$.

$\frac{dy}{dx} = 2x - 2 - 3x^{-2}$

at $x = 2$: $2(2) - 2 - \frac{3}{2^2}$

$= 2 - \frac{3}{4}$

$= \boxed{1.25}$

Finding the Gradient of the Curve

• The gradient of the curve of a function at a specific point = gradient of the tangent at that point.

∴ use same method (same thing, different way of asking)

1. differentiate the function.
2. substitute the x-value of the point into the derivative.

Worked Examples

6. Find the gradient of the tangent to the curve whose equation is $y = \frac{4}{x} + 2$ at the point where $x = 2$

$y = 4x^{-1} + 2$

$\frac{dy}{dx} = -4x^{-2}$

@ $x = 2$ grad $= \frac{-4}{2^2}$

$= \boxed{-1}$

7. A power boat moves in a straight line such that at time t seconds its distance s from a fixed point O on that line, is given by the equation $\boxed{s = 2t^2 - 3t + 1.}$ Find the speed after 3 seconds.

gradient of distance/time curve

$\frac{ds}{dt} = 4t - 3$

at $t = 3$ speed $= 4(3) - 3$

$= \boxed{9 \text{ m/s}}$

Examples to Try

8. Find the value of gradient of the curve whose equation is $y = (x-3)(x+2)$ at the point where it crosses the positive x-axis.

roots @ $x=3$, $x=-2$

at $x=3$

a) expand, then differentiate

$y = x^2 - x - 6$

$\frac{dy}{dx} = 2x - 1$

gradient @ $x=3$ $= 2(3) - 1$

$= \boxed{5}$

9. The equation of a curve is given by $y = 5x^3 - 2x^2 - 4$. Find the x-coordinates of the points where the gradient is 5.

given gradient, find x

$\frac{dy}{dx} = 15x^2 - 4x$

grad → $5 = 15x^2 - 4x$

$0 = 15x^2 - 4x - 5$ ← Quadratic solve in GDC

$\boxed{x = 0.726,\ = -0.459}$

Examples to try

10. Find a if $f(x) = ax^2 + 6x - 3$ has a tangent with gradient 2 at the point where $x = -1$.

$y = ax^2 + 6x - 3$

$\frac{dy}{dx} = 2ax + 6$

$2 = 2a(-1) + 6$

$-4 = -2a$

$\boxed{2 = a}$

Finding a Point on f(x) given the Gradient

- If you are given the gradient of a curve, but not the point, you can work backwards to find the coordinates of the point.

1. differentiate the function.
2. Make f'(x) = gradient given
3. Solve the equation to find x
4. Substitute x into f(x) to find the y value (if need co-ordinate)

original

Worked Examples

11. Point A lies on the curve $y = 4x^2 + 4x$ and the gradient at A is 16.
Find the coordinates of A.

slope

$\frac{dy}{dx} = 8x + 4$

$16 = 8x + 4$

$12 = 8x$

$\frac{12}{8} = \boxed{x = \frac{3}{2}}$

$A = \left(\frac{3}{2}, 15\right)$

find y value (original function)

$y = 4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right)$

$= \frac{\cancel{4}(9)}{\cancel{4}} + \frac{\cancel{12}\,6}{\cancel{2}} = \boxed{15}$

12. Point A lies on the curve $y = 4x^2 - 5x + 8$ and the gradient at A is -21.
Find the coordinates of A.

$\frac{dy}{dx} = 8x - 5$

$-21 = 8x - 5$

$-16 = 8x$

$\boxed{-2 = x}$

$y = 4(-2)^2 - 5(-2) + 8$

$= 16 + 10 + 8$

$= 34$

$(-2, 34)$

Homework

Exercise 6 A p254: Q1-4, (2nd column)

Exercise 6B p265: Q2, b, d, f

Exercise 6C p266: Q 3, 6, 9, 12, 15

Exercise 6D p267: do any 5 (use GDC)

Exercise 6E p269: Q 5, 7, 8