Introductory Differential Calculus

2: Equations of Tangents & Normals & Graph Shapes

Chapter 6, p271 – 283

Finding the Equation of a Tangent

1. Find the slope of the tangent by differentiating
2. Write the equation of a line $y = mx + c$ using the slope.
3. Substitute (x, y) of the point the tangent touches the function and solve for c.
4. Write the final equation of the line.

Worked Examples

1. Find the equation of the tangent to the curve whose equation is $y = 3x^2 + \frac{2}{x} - 5$ at the point P where $x = 2$.

$y = 3x^2 + 2x^{-1} - 5$

a) differentiate: $\frac{dy}{dx} = 6x - 2x^{-2}$

b) slope of tangent @ $x = 2$

slope $= 6(2) - \frac{2}{2^2}$

$= 12 - \frac{1}{2}$

$= 11\frac{1}{2}$

c) find y at $x=2$ original function

$y = 3(2)^2 + \frac{2}{2} - 5$

$= 8$

$\therefore (2, 8)$

equation of tangent:

$y = 11.5x + c$

$8 = 11.5(2) + c$

$8 = 23 + c$

$-15 = c$

$\boxed{y = 11.5x - 15}$

Using the Calculator – Ti 84

Find the equation of the tangent to a curve:

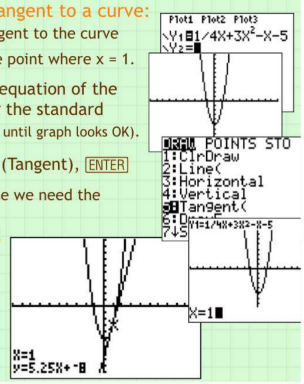
eg Find the equation of the tangent to the curve $y = \frac{1}{4}x^4 + 3x^2 - x - 5$ at the point where x = 1.

- Press Y= and type in the equation of the curve. Press ZOOM 6 for the standard window. (Adjust the window until graph looks OK).
- Press 2nd PRGM (DRAW) 5 (Tangent), ENTER

Press 1 ENTER (press 1 because we need the tangent at x = 1).

The tangent line is drawn at x=1 and the equation of the tangent is given.

NOTE: to remove the tangent (eg to find the tangent at another point) 2nd PRGM 1 Clear Draw

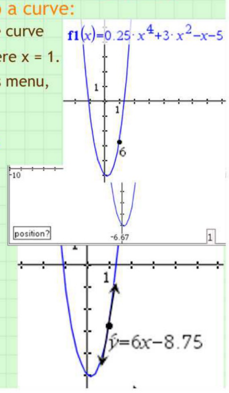


Using the Calculator – Nspire

Find the equation of the tangent to a curve:

eg Find the equation of the tangent to the curve $y = \frac{1}{4}x^4 + 3x^2 - x - 5$ at the point where x = 1.

- Open **Geometry page** from home. Press menu, 1: Graphing, enter function.
- Press menu, 6: Analyse graph, 5:dy/dx. Type in x value, enter. The slope of the curve is displayed next to a dot on the function at the x-value
- Press menu, 8: Geometry, 1:Points & Lines, 7: Tangent. Move cursor to point at dot ("point on tab"), press enter. The tangent line is displayed and the equation of the tangent is displayed next to it.
- Check that the slope of the tangent is equal to the slope of the graph displayed earlier.



Worked Examples

2. For the function $y = 1 - 3x + 12x^2 - 8x^3$

a) find $\frac{dy}{dx}$

b) find the equation of the tangent at (1, 2)

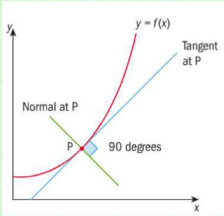
$\boxed{y = -3x + 5}$

3. Find a if $f(x) = ax^2 + 6x - 3$ has a tangent with gradient 2 at the point where $x = -1$.

$\boxed{a = 2}$

Normal to a Curve

- The normal to the curve is a straight line which is perpendicular to the tangent.
- The normal and the tangent intersect at the point where the tangent touches the curve.
- The gradient of the normal is the negative reciprical of the gradient of the tangent.



Finding the Equation of the Normal

1. Find the slope of the tangent by differentiating
2. Take the negative reciprical of the slope (= slope of the normal)
3. Write the equation of a line $y = mx + c$ using the slope.
4. Substitute (x, y) of the point the tangent/normal touches the function and solve for c.
5. Write the final equation of the line.

Worked Examples

4. Find the equation of the normal to the function $y = x^3 + x + 2$ at the point $x = 1$.

find y when $x = 1$ $\rightarrow$ $y = 1^3 + 1 + 2 = 4$

$(1, 4)$

$\frac{dy}{dx} = 3x^2 + 1$

slope of tangent at $x = 1$: $3(1)^2 + 1 = 4$

slope of normal $= -\frac{1}{4}$

Equation of normal

$y = -\frac{1}{4}x + c$

$4 = -\frac{1}{4}(1) + c$

$\frac{16}{4} + \frac{1}{4} = c$ $\quad c = \frac{17}{4}$

$\boxed{y = -\frac{1}{4}x + \frac{17}{4}}$

Example to Try

5. Find the equation of the normal to the function $y = 12x^{-1}$ at the point $x = -4$.
Give your answer in the form $ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$.

$y = \frac{12}{-4} = -3$

$\frac{dy}{dx} = -12x^{-2}$

slope of tangent @ $x = -4$ $= \frac{-12}{(-4)^2} = -\frac{12}{16} = -\frac{3}{4}$

slope of normal $= \frac{4}{3}$

Equation of normal.

$y = \frac{4}{3}x + c$

$-3 = \frac{4}{3}(-4) + c$

$-\frac{9}{3} + \frac{16}{3} = c$

$c = \frac{7}{3}$

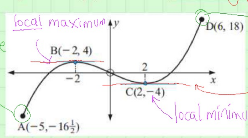
$\boxed{y = \frac{4}{3}x + \frac{7}{3}}$

$3y = 4x + 7$

$\boxed{4x - 3y + 7 = 0}$

Horizontal Tangents

- Where the function has a local minimum, maximum or inflection, the tangent will be a horizontal line.
- The slope of the tangent is zero.
- By making the derivate = 0 and solving for x we can find the position(s) of these turning points.



Worked Examples

6. Find the coordinates of any points on the curve with the equation $f(x) = x^3 + 3x^2 - 9x + 5$ where the tangent is horizontal.

make derivate = 0

$f'(x) = 3x^2 + 6x - 9$

slope of tangent = 0

$3x^2 + 6x - 9 = 0$ solve for x

$x = -3 \quad x = 1$ GDC – polynomial tools

need coordinates $\rightarrow$ substitute into original function

$y = (-3)^3 + 3(-3)^2 - 9(-3) + 5$
$= -27 + 27 + 27 + 5$
$= 32$

$\boxed{(-3, 32)}$

$y = (1)^3 + 3(1)^2 - 9(1) + 5$
$= 1 + 3 - 9 + 5$
$= 0$

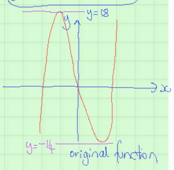
$\boxed{(1, 0)}$

Example to Try

7. Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

$y = \#$, Slope $= 0$

$f'(x) = 0$



$\frac{dy}{dx} = 3x^2 - 12$

horizontal tangent @ $3x^2 - 12 = 0$

$x = 2 \quad x = -2$

substitute into original function → y values

$y = (2)^3 - 12(2) + 2$
$= 8 - 24 + 2$
$= -14$

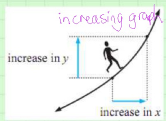
$\therefore \boxed{y = -14}$

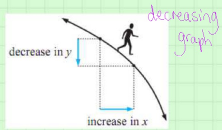
$y = (-2)^3 - 12(-2) + 2$
$= -8 + 24 + 2$
$= 18$

$\therefore \boxed{y = 18}$

Graph Shapes

- By looking at the graph of a function, we can easily see if the graph is increasing (positive slope of tangents) Or decreasing (negative slope of tangents)
- An increasing function is where an increase in x produces an increase in y.
- A decreasing function is where an increase in x produce a decrease in y.

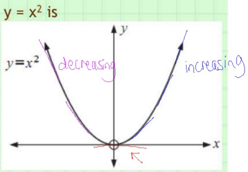




Graph Shapes

- Most functions have intervals where the graph is increasing an other areas where it is decreasing.
- "turning points" or maximum and minimums are points on the graph where the direction of the graph changes.
- For example: Find intervals where $y = x^2$ is
  - a) increasing $x > 0$
  - b) decreasing $x < 0$

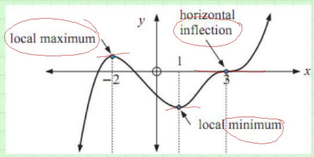
at $x = 0$, the graph is stationary (horizontal tangent)



Stationary Points

- At a stationary point on a curve, the slope of the tangent to the curve is equal to zero.
- Stationary points can be:
  - a local maximum (curve goes from inc to dec)
  - a local minimum (curve goes from dec to increasing)
  - an inflection (curve keeps same direction)

inflection



NB inflection are not in syllabus

Worked Examples

8. Find and classify all stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$

$f'(x) = 0$

Step 1: Differentiate function to solve for $f'(x) = 0$

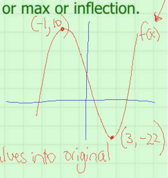
Step 2: Put x values back into original function f(x) to find y values

Step 3: plot graph on calculator to find out if min or max or inflection.

$f'(x) = 3x^2 - 6x - 9$

make $f'(x) = 0$ $\quad 3x^2 - 6x - 9 = 0$

$\boxed{x = 3 \quad x = -1}$ (GDC)



Graph original function OR substitute x values into original

$(-1, 10) \rightarrow$ maximum

$(3, -22) \rightarrow$ minimum

Worked Examples

9. $f(x) = x^3 + ax + b$ has a stationary point at (1, 8).

a) Find the values of a and b $\quad f'(x) = 0$

b) Find the position and nature of all stationary points

Coordinates $\quad$ min/max

① $f'(x) = 3x^2 + a$

② $f'(1) \Rightarrow 3x^2 + a = 0$

$3(1)^2 + a = 0$

$\boxed{a = -3}$

sub into f(x)

③ $8 = (1)^3 - 3(1) + b$

$8 = 1 - 3 + b$

$\boxed{b = 10}$

$\boxed{f(x) = x^3 - 3x + 10}$

need to find the other stationary pt

④ Find other stationary point

$3x^2 - 3 = 0$ ← GDC poly tools

$3x^2 = 3$

$x^2 = 1 \quad \boxed{x = \pm 1}$

$y = (-1)^3 - 3(-1) + 10$

$= -1 + 3 + 10 = 12$

$(-1, 12)$

Now: Graph original function

$\boxed{(1, 8) = \text{min} \quad (-1, 12) = \text{max}}$

Worked Examples

10. The table below describes the behaviour of f′(x), the derivative function of f(x) in the domain 0 ≤ x ≤ 11.

a) Sketch the graph of f(x) using the information given.

x	f′(x)
0 ≤ x < 3	> 0 +
3	0
3 < x < 5	<0 −
5	0
5 < x < 9	>0 +
9	0
9 < x ≤ 11	<0 −

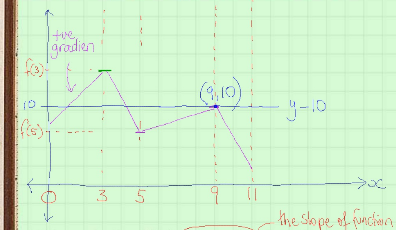
slope of tangent

horizontal line

Steps

set up scale (x-axis)

2 in pencil, mark boundaries



the slope of function bet x=3 & x=5 is negative

b) State whether f(3) is greater than, less than or equal to f(5). Give a reason. (2)

c) The point (9, 10) lies on the graph, give the equation of the tangent at this point. (1 or 2)

d) From the info given, state whether (9, 10) is a maximum, minimum or neither. (2)

local maximum

Homework

Exercise 6 F p272: Q1 (do any 4), Q2 (any 2) tangents

Exercise 6 G p273: do any 4 normals

more exam style Qs ← Exercise 6 H p274: Q1, 11, 15 tangents & normals

Exercise 6 J p281: do any 4

Exercise 6 K, 6L p282: do any 3 from each