Introductory Differential Calculus

3: Rates of Change & Optimisation

Chapter 6, pages 275-279, 283-290

Rates of Change

- The derivate of a function measures how the y values change as x increases (in the original function).
- For a function y = f(x), the derivative f'(x) gives the rate of change of y with respect to x.
  "slope"
- Often the x-value will be time, so the derivate measures rate of change over time.

* the derivative of distance = speed * the derivative of speed = acceleration

Worked Examples

1. A toy car moves in a straight line with a displacement of *s* metres *t* minutes after leaving a fixed point. The displacement is given by the formula

distance position $\boxed{s(t) = 2t^3 - 21t^2 + 60t + 3,}$ for $t \geq 0$

$\boxed{S = \frac{d}{t} \quad d = st}$

a) What does $\frac{ds}{dt}$ represent? change in displacement over time
"speed" or rate of change of displacement with time

b) What are the units for $\frac{ds}{dt}$? $\frac{\text{metres}}{\text{minutes}}$ = m/minutes

c) Find the initial position of the car? $s(0) = 2(0)^3 - 21(0) + 60(0) + 3$
t=0
$= \boxed{3 \text{ metres from point}}$

d) Find the value of $\frac{ds}{dt}$ at t = 0 $\frac{ds}{dt} = 6t^2 - 42t + 60$
speed time
at $t = 0$ $\frac{ds}{dt} = 60$ metres/minutes

e) Find when the car was not moving
@ speed = 0 ∴ $\frac{ds}{dt} = 0$ $6t^2 - 42t + 60 = 0$ t = 2 mins, 5 mins

f) Describe the motion of the car for the first 7 minutes
moved away for 2 mins stopped, moved towards the point for 3 mins, stopped & moved away for 2 mins
original function
2 5

Example to Try

2. A factory is producing mathematical puzzles. The cost (in dollars) of producing *n* units of the puzzle per day is modeled by the function

$\boxed{C(n) = -0.05\, n^2 + 10\, n + 50}$

Cost
\# of units

a) What does $\frac{dC}{dn}$ represent? (change in) cost per unit
rate of change of

b) Find $\boxed{\frac{dC}{dn}} = \boxed{-0.1n + 10}$

c) Find the value of C and $\frac{dC}{dn}$ when i) $\boxed{n = 10}$ and ii) $\boxed{n = 50}$
@ n = 10 Cost = $-0.05(10)^2 + 10(10) + 50$ = \$145
$\frac{dc}{dn} = -0.1(10) + 10 = 9$ \$9/unit
@ n = 50 Cost = $-0.05(50)^2 + 10(50) + 50$ = \$425 $\frac{dc}{dn} = -0.1(50) + 10 = 5$
\$5/unit

d) Interpret your answers to part c.
making 50 units a day is cheaper cost/unit although total costs inc

Max/Min Problems

- Many problems where we try to find the minimum or maximum value of a variable can be solved using calculus.
- The solution is often referred to as the optimum.
  best possible

Optimisation Problem Solving Method

1. Draw a large, clear diagram of the situation.
2. Make an equation with the variable that needs to be a maximum or minimum, as the subject of the formula in terms of one convenient variable (x say). Also find what restrictions there may be on x.
3. Find f'(x) and find the value(s) of x when f'(x) = 0.

Optimisation Problem Solving Method

4. If there is a restricted domain such as $a \leq x \leq b$, the max/min of the function may occur either when $f'(x) = 0$ or at $x = a$ or $x = b$.

5. Draw a sketch from your calculator and label main points clearly.

Worked Examples

3. Two numbers have a sum of 10. What is the maximum value of their product?

x and y

$x = 5$

$x + y = 10$

$y = 10 - x$ * rearrange so you only have 1 variable

their product $= x(10 - x)$ graph

$P = 10x - x^2$ ← equation with product as the subject

To find max ∴ differentiate the equation

$\frac{dP}{dx} = 10 - 2x$ ← make $= 0$

$10 - 2x = 0$

$10 = 2x$

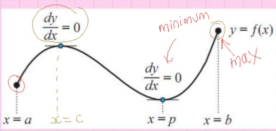
$\boxed{5 = x}$

∴ max product $= 5 \times 5 = \boxed{25}$

Optimisation Problem Solving Method

• Warning: The maximum/minimum value isn't always when $f'(x) = 0$. Check the entire graph carefully for the overall minimum and maximum values.

In this example, the minimum is at $x = p$, but the maximum is at $x = b$.

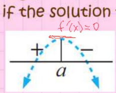


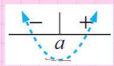
• Always check the values of the function at the endpoints of the domain to see if they are the minimum or maximum values.

Testing Optimal Solutions

• If you find values of x for $f'(x) = 0$ when solving these problems, how will you know if you have a maximum or minimum?

• Find increasing/decreasing Regions:
Substitute values on either side of x in $f'(x)$. If the value is positive, the graph is increasing, if the value is negative, the graph is decreasing.
Eg if the solution to $f'(x) = 0$ is $x = a$,





• Graphical Test: *
Graph the original equation, ∩ is a maximum, and ∪ is a minimum.

Worked Examples

4. The student council is planning a school dance in the gymnasium and has to define the dance floor so people know where to boogie. One side of the dance floor is against the wall and the other three sides will be roped off. If the council has 20 metres of rope, what should the dimensions of the dance floor be in order to maximise the area?

x 5m, 10m, y, $20 - 2x$, x 5m, wall

$20 = 2x + y$

$y = 20 - 2x$

Area $= x(20 - 2x)$

$\boxed{A = 20x - 2x^2}$

$A'(x) = 20 - 4x$

$20 - 4x = 0$

$20 = 4x$

$\boxed{5 = x}$

Dance floor = 5m x 10m

Example to Try

5. Ravi knows that the product of two positive numbers, x and y, is 200.

a) Find a formula for their sum, S, in terms of x only.

b) Use calculus to find the ~~minimum~~ maximum value of S.

S = 200x - x²

S'(x) = 200 - 2x

200 - 2x = 0 (to find max)

x = 100

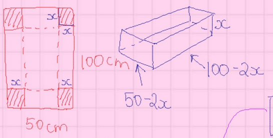
Max value of S = 100(100) = 10000

Worked Examples

6. A sheet of then card 50 cm by 100 cm has a square of side x cm cut away from each corner and the sides folded up to make a rectangular open box.

   a) Find the volume, V, of the box in terms of x. $V = \_\_ x$

   b) Using calculus, find the value of x which gives a maximum value of the box.

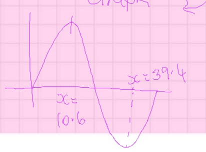


$$\boxed{V = x(50-2x)(100-2x)}$$

$$= x(5000 - 100x - 200x + 4x^2)$$

$$\boxed{V = 5000x - 300x^2 + 4x^3}$$

Graph



$$\frac{dV}{dx} = 5000 - 600x + 12x^2$$

$$0 = 12x^2 - 600x + 5000$$

$$\boxed{x = 10.566} \quad x = 39.43$$

max value

Example to Try

7. At time t seconds after a ball is thrown vertically upwards, its height, h in meters, above the point of projection is given by the formula $h = 18t - 5t^2$.

   a) Find the time when the ball is at its maximum height.

   b) Find the maximum height.

$h'(t) = 18 - 10t$

$18 - 10t = 0$ (to find max)

$t = 1.8$ seconds

substitute into original formula:

$h = 18(1.8) - 5(1.8)^2$

max height = 16.2 meters

Note - thsi question can be solved as a quadratic word question without using calculus - the max is at the vertex on the axis of symmetry.

Homework

Exercise 6 I p267: do at least 4

Exercise 6M p285: Q 1 – 4;

Exercise 6N p288: Q 1, 3, 4, 9