Logic

2: Arguments

p395 – 408

Logical Equivalence

- If two compound propositions have the same T/F column, we say that they are logically equivalent (logically the same).

Eg: Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Truth table for $(p \wedge q)$:

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Truth table for $p \vee q$:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

same

Tautologies & Logical Contradictions

- A tautologies is a compound statement which is always true (true for all possibilities in the truth table).
- A logical contradiction is a compound statement which is always false (False for all possibilities in the truth table).

Eg: Show that $p \vee \neg p$ is a tautology:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

all Ts ∴ tautology

Worked Examples

1. Show that $(\neg p \vee q) \wedge (\neg q \wedge p)$ is a logical contradiction.

p	q	¬p	¬q	¬p∨q	¬q∧p	(¬p∨q)∧(¬q∧p)
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

Worked Examples

2. Show that $(p \wedge q) \vee (\neg p \vee \neg q)$ is a tautology.

p	q	¬p	¬q	p∧q	¬p∨¬q	(p∧q)∨(¬p∨¬q)
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

all true ∴ tautology

Implication = If ....Then

- A compound statement saying "if.....then.....", is called an implication.
- Symbol = $\Rightarrow$
- The order of the propositions is important & sometimes not all combinations in the truth table are possible.
- Truth table:
- note that the implication is only false when p is T & q is F

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Worked Examples

4. Show that the proposition $(p \veebar q) \vee (p \wedge q) \Rightarrow (p \vee q)$ is a tautology.

F: T⇒F

p	q	p ⊻ q	p ∧ q	(p ⊻ q) ∨ (p ∧ q)	p ∨ q	((p ⊻ q) ∨ (p ∧ q)) ⇒ p ∨ q
T	T	F	T	T	T	T
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	F	F	T

all Ts tautology

Equivalence = If & only if

- Two statements are equivalent if one implies the other and vise versa.
- Symbol = ⇔
- $p \Leftrightarrow q$ is the conjunction of 2 implications: $p \Rightarrow q$ & $q \Rightarrow p$
  so $p \Leftrightarrow q$ = $(p \Rightarrow q) \wedge (q \Rightarrow p)$
- We say "if and only if" when talking about equivalence.

Eg: for p: "I will pass the exam" and q: "the exam is easy"

- $p \Rightarrow q$ *If* I pass the exam, *then* the exam is easy
- $q \Rightarrow p$ *If* the exam is easy, *then* I will pass the exam
- $p \Leftrightarrow q$ I pass the exam, *if and only if* the exam is easy

Equivalence = If & only if

- Truth table:

p⇔q

p	q	p ⇒ q	∧	q ⇒ p
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

- Truth table for equivalence:

2 same = T

p	q	p ⇔ q
T	T	T
T	F	F
F	T	F
F	F	T

Worked Examples

5. Suppose that p represents "it is raining" and q represents "there are puddles forming". *Write the following statements in symbols.*

a) If it is raining, then puddles are forming $p \Rightarrow q$

b) If puddles are forming, then it is raining $q \Rightarrow p$

c) Puddles are not forming $\neg q$

d) It is not raining $\neg p$

e) If it is not raining, then puddles are not forming $\neg p \Rightarrow \neg q$

f) Puddles are forming if and only if it is raining $q \Leftrightarrow p$

g) If it is raining, then puddles are not forming $p \Rightarrow \neg q$

Worked Examples

T⇒F F

6. Construct truth tables for:

a) $(p \wedge q) \Rightarrow p$

b) $p \Leftrightarrow \neg q$

c) $(p \Leftrightarrow q) \wedge \neg p$

d) $(p \Rightarrow q) \Rightarrow \neg p$

p	q	¬p	¬q	p ∧ q	(p ∧ q) ⇒ p (a)	p ⇔ ¬q (b)	p ⇔ q	(p ⇔ q) ∧ ¬p (c)	p ⇒ q	(p ⇒ q) ⇒ ¬p (d)
T	T	F	F	T	T	F	T	F	T	F
T	F	F	T	F	T	T	F	F	F	T
F	T	T	F	F	T	T	F	F	T	T
F	F	T	T	F	T	F	T	T	T	T

Valid Arguments

- An argument is made up of premises (propositions) that lead to a conclusion.
- Arguments are compound statements with an implication.
- Arguments are "valid" if they are always true (tautology).

Eg: Determine the validity of this argument: $(p \wedge q) \Rightarrow p$.

p	q	p ∧ q	(p ∧ q) ⇒ p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

tautology ∴ argument is valid

Worked Examples

2. Determine the validity of this argument: $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \Leftrightarrow q)$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$	$\Rightarrow$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T
		①	①	①	①	①

The argument is valid because it is a tautology. ①

Worked Examples (IB Question)

7. Write down the values for **a**, **b**, **c**, **d**, **e** and **f** from the table below: (*6 marks*)

p	q	p	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F **a**			F **d**		
T	F		F **b**				F **f**
F	T			T **c**			
F	F					T **e**	

Homework

Exercise 9F p397, Q 4(all);
9G p398, odds;
9H p400, Q 8; 9I p403, Q 3, 5, 7;
9K p408 Q6, 7, 8